An Analysis of Quicksort: Average Case
Assumptions

- Average will be taken over Location of Pivot
- All Pivot Positions are equally likely
- Pivot positions in each call are independent of one another
Formulation 1

- $A(0) = 0$
- If the pivot appears at position $i$, $1 \leq i \leq n$ then $A(i-1)$ comparisons are done on the left hand list and $A(n-i)$ are done on the right hand list.
- $n-1$ comparisons are needed to split the list.
Recurrence Relation

\[ A(n) = n - 1 + \frac{1}{n} \sum_{i=1}^{n} (A(i - 1) + A(n - i)) \]

\[ \sum_{i=1}^{n} (A(i - 1) + A(n - i)) \]

\[ = (A(0) + A(n - 1)) + (A(1) + A(n - 2)) + \cdots + (A((n - 1) - 1) + A(n - (n - 1))) + (A(n - 1) + A(n - n)) \]
Solve the Recurrence

\[\sum_{i=1}^{n} (A(i - 1) + A(n - i)) = 2A(0) + 2\sum_{i=1}^{n-1} A(i)\]

\[A(n) = \Theta(n) + \frac{2}{n} \sum_{i=1}^{n-1} A(i)\]
Guessing the Solution

Guess: \( A(n) \leq an \lg n + b \quad a>0, \ b>0 \)

\[
A(n) = \Theta(n) + \frac{2}{n} \sum_{i=1}^{n-1} A(i)
\]

\[
\leq \Theta(n) + \frac{2}{n} \sum_{i=1}^{n-1} (ai \lg i + b)
\]

\[
= \Theta(n) + \frac{2a}{n} \sum_{i=1}^{n-1} i \lg i + \frac{2b}{n} (n - 1)
\]
By Integration:

\[
\sum_{i=1}^{n-1} i \lg i \leq \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2
\]
And Finally ...

\[
A(n) \leq \frac{2a}{n} \left( \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2 \right) + \frac{2b}{n} (n - 1) + \Theta(n)
\]

\[
\leq an \lg n - \frac{a}{4} n + 2b + \Theta(n)
\]

\[
= an \lg n - \frac{a}{4} n + 2b + \Theta(n)
\]

\[
\leq an \lg n + b
\]
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\[ \sum_{i=1}^{n} (A(i - 1) + A(n - i)) = 2A(0) + 2 \sum_{i=1}^{n-1} A(i) \]

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Guessing the Solution

**Guess:** $A(n) \leq an \lg n + b \quad a > 0, \ b > 0$

$$A(n) = \Theta(n) + \frac{2}{n} \sum_{i=1}^{n-1} A(i)$$

$$\leq \Theta(n) + \frac{2}{n} \sum_{i=1}^{n-1} (ai \lg i + b)$$

$$= \Theta(n) + \frac{2a}{n} \sum_{i=1}^{n-1} i \lg i + \frac{2b}{n} (n - 1)$$
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And Finally ...

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