# *An Analysis of Quicksort:*

## **Average Case**

# *Assumptions*

- **Average will be taken over *Location of Pivot***
- **All Pivot Positions are equally likely**
- **Pivot positions in each call are independent of one another**

# Formulation I

- **$A(0) = 0$**
- **If the pivot appears at position i, $1 \le i \le n$ then $A(i-1)$ comparisons are done on the left hand list and $A(n-i)$ are done on the right hand list.**
- **n-1 comparisons are needed to split the list**

# Recurrence Relation

$$A(n) = n - 1 + \frac{1}{n}\sum_{i=1}^{n}\big(A(i-1) + A(n-i)\big)$$

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$$\sum_{i=1}^{n}\big(A(i-1) + A(n-i)\big)$$

$$= \big(A(0) + A(n-1)\big) + \big(A(1) + A(n-2)\big) + \cdots + \big(A((n-1)-1) + A(n-(n-1))\big) + \big(A(n-1) + A(n-n)\big)$$

# Solve the Recurrence

$$\sum_{i=1}^{n} \left( A(i-1) + A(n-i) \right) =$$

$$2A(0) + 2\sum_{i=1}^{n-1} A(i)$$

$$A(n) = \Theta(n) + \frac{2}{n}\sum_{i=1}^{n-1} A(i)$$

# Guessing the Solution

**Guess:** $A(n) \le an \lg n + b \qquad a>0,\ b>0$

$$A(n) = \Theta(n) + \frac{2}{n}\sum_{i=1}^{n-1} A(i)$$

$$\le \Theta(n) + \frac{2}{n}\sum_{i=1}^{n-1}(ai \lg i + b)$$

$$= \Theta(n) + \frac{2a}{n}\sum_{i=1}^{n-1} i \lg i + \frac{2b}{n}(n-1)$$

# Continuing …

**By Integration:**

$$\sum_{i=1}^{n-1} i \lg i \leq \frac{1}{2} n^2 \lg n - \frac{1}{8} n^2$$

# And Finally ...

$$A(n) \le \frac{2a}{n}\left(\frac{1}{2}n^2 \lg n - \frac{1}{8}n^2\right) + \frac{2b}{n}(n-1) + \Theta(n)$$

$$\le an\lg n - \frac{a}{4}n + 2b + \Theta(n)$$

$$= an\lg n - \frac{a}{4}n + 2b + \Theta(n)$$

$$\le an\lg n + b$$

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$$\cdots + \left(A((n-1)-1) + A(n-(n-1))\right) +$$

$$\left(A(n-1) + A(n-n)\right)$$

# Solve the Recurrence

$$\sum_{i=1}^{n}\left(A(i-1)+A(n-i)\right)=$$

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$$A(n)=\Theta(n)+\frac{2}{n}\sum_{i=1}^{n-1}A(i)$$

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# And Finally …

$$
\begin{aligned}
A(n) &\le \frac{2a}{n}\left(\frac{1}{2}n^2 \lg n - \frac{1}{8}n^2\right) + \frac{2b}{n}(n-1) + \Theta(n) \\
&\le an \lg n - \frac{a}{4}n + 2b + \Theta(n) \\
&= an \lg n - \frac{a}{4}n + 2b + \Theta(n) \\
&\le an \lg n + b
\end{aligned}
$$