Recurrence Relations

Analyzing the performance of recursive algorithms

Worst Case Analysis

- Select a Hypothetical Set of Inputs
- Should be expandable to any size
- Make Algorithm run as long as possible
- Must not depend on Algorithm Behavior
  - May use static behavior analysis
  - Inputs may not change dynamically

Average Case Analysis

- Determine the range of inputs over which the average will be taken
- Select a probability distribution
- Characterize Amount of work required for each input
- Compute the Mean Value

Binary Search

```
BinS(X:Item;First,Last:Integer):Integer
var Middle:Integer;
begin
    if First <= Last then
        Middle = (First + Last)/2;
        if L[Middle] = X then return Middle;
        if L[Middle] < X then return BinS(X,Middle+1,Last);
        if L[Middle] > X then return BinS(X,First,Middle-1);
    else
        return -1;
    endif;
end;
```

Analysis of BinS: Basis

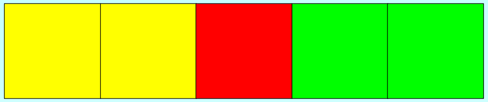
- If List Size = 1 then First = Last
- If List Size = 0 then First > Last

  if First <= Last then ...

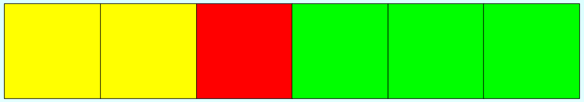
  else   return -1;
- Then W(0) = 0

Analysis of BinS: Recurrsion I

- If comparison fails, find size of new list
- If list size is odd:
  - First=1,Last=5,Middle=(1+5)/2=3



- If list size is even:
  - First=1,Last=6,Middle=(1+6)/2=3



Analysis of BinS: Recurrsion II

- Exercise: Verify that these examples are characteristic of *all* lists of odd or even size
- Worst case: Item X not in list, larger than any element in list.
- If list is of size n, new list will be of size:

$$\left\lfloor \frac{n}{2} \right\rfloor$$

Analysis of BinS: Recurrence

$$W(0) = 0$$

$$W(n) = 1 + W(\lfloor n/2 \rfloor)$$

Is $W(n) \in \Theta(n)$?

Solving the Recurrence

$$W(n) = 1 + \boxed{W(\lfloor n/2 \rfloor)}$$

We need to eliminate W from the RHS.

$$W(\lfloor n/2 \rfloor) = 1 + W(\lfloor \lfloor n/2 \rfloor / 2 \rfloor)$$

$$= \boxed{1 + W(\lfloor n/4 \rfloor)}$$

$$W(n) = 1 + \boxed{1 + W(\lfloor n/4 \rfloor)}$$

Continuing the Exercise ...

$$W(n) = 1 + W(\lfloor n/2 \rfloor)$$

$$W(n) = 1 + 1 + W(\lfloor n/4 \rfloor)$$

$$W(n) = 1 + 1 + 1 + W(\lfloor n/8 \rfloor)$$

$$W(n) = 1 + 1 + 1 + 1 + W(\lfloor n/16 \rfloor)$$

Ah Ha! A General Formula!

$$W(n) = k + W\left(\left\lfloor n / 2^k \right\rfloor\right)$$

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***Since* W(0) = 0 ...**

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But, if k is big enough ...

Then the argument of W will be zero, and ...

Since W(0) = 0 ...

The W on the RHS ...

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$$W(n) = k + W\left(\lfloor n / 2^k \rfloor\right)$$

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Will Disappear!

When Will W disappear?

When:

$$2^{k-1} \leq n < 2^k$$

When Will W Disappear?

When:

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Take the Log:

$$k - 1 \le \lg n < k$$

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When:

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Take the Log:

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Take the Floor:

$$k-1 = \lfloor \lg n \rfloor$$

$$k = \lfloor \lg n \rfloor + 1$$

Finally ...

$$W(n) = k + W\left(\left\lfloor n / 2^k \right\rfloor\right)$$

$$W(n) = \lfloor \lg n \rfloor + 1 + W(0)$$

$$W(n) = \lfloor \lg n \rfloor + 1$$

Some Other Recurrences

$$Q(n) = n - 1 + 2\,Q(n/2)$$
$$Q(1) = 0$$

$$W(n) = cn + W(n/2)$$
$$W(1) = 1$$

Recurrences with T(1)=1

$$\mathrm{T}(n) = \mathrm{T}(n/2) + c\lg n$$

$$\mathrm{T}(n) = \mathrm{T}(n/2) + cn$$

$$\mathrm{T}(n) = 2\,\mathrm{T}(n/2) + cn$$

$$\mathrm{T}(n) = 2\,\mathrm{T}(n/2) + cn\lg n$$

$$\mathrm{T}(n) = 2\,\mathrm{T}(n/2) + cn^2$$

The Challenger

$$T(n) = \sqrt{n}\,T\left(\sqrt{n}\right) + cn$$

$$T(2) = 1$$

The Master Theorem I

$$T(n) = a\,T(n/b) + f(n)$$

$$\text{if } f(n) \in O\left(n^{\log_b a - e}\right), \text{ for some } e > 0$$

$$T(n) \in \Theta(n^{\log_b a})$$

The Master Theorem II

$$T(n) = a\,T(n/b) + f(n)$$

$$\text{if } f(n) \in \Theta\left(n^{\log_b a}\right)$$

$$T(n) \in \Theta(n^{\log_b a} \lg n)$$

The Master Theorem III

$$\mathrm{T}(n) = a\,\mathrm{T}(n/b) + f(n)$$

$$\text{if } f(n) \in \Omega\left(n^{\log_b a + e}\right), \text{ for some } e > 0$$

$$\text{and } af(n/b) \le cf(n) \text{ for some } c < 1$$

(for sufficiently large n)

$$\mathrm{T}(n) \in \Theta(f(n))$$

Homework

- Page 72, 4-1
- Page 73, 4-4

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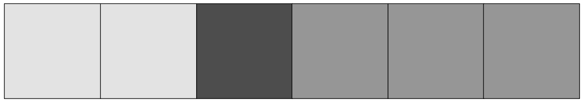

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