Free Beer!
All Functions Please
Line Up
In Descending Order
By Growth Rate!

Solutions and Notes
For Problem 2-3a
by Peter M. Maurer
MY SOLUTION TO 2-3A

\[ 2^{2n+1} = 2^{2n} = \left(2^n\right)^2 \]

\[ 2^{2n} \]

\[ (n + 1)! \]

\[ n! \]

\[ e^n \]

\[ n \cdot 2^n \]

\[ 2^n \]

\[ \left(\frac{3}{2}\right)^n \]

\[ n^{\log_{2} n} = \left(\log_{2} n\right)^n = n^{\log_{n} n} \]

\[ (\log n)! \]

\[ n^3 \]

\[ \frac{n^2 + 1}{4} = n^2 \]

\[ n \log n \]

\[ \log(n!) \in \Theta(n \log n) \]

\[ n \]

\[ 2^{\log n} = n \]

\[ \left(\sqrt{2}\right)^{\log n} = 2^{\log_{2} n} = \left(2^{\log_{2} n}\right)^5 = \sqrt{n} \]

\[ 2^{\sqrt{2} \log n} = n^{\sqrt{2} / \log n} \]

\[ \log^{2} n \]

\[ \ln n \]

\[ \sqrt{\log n} \]

\[ \ln \ln n \]

\[ \sqrt{2} \log^{*} n \]

\[ \log^{*} n \]

\[ \log^{*} (\log n) = \log^{*} n - 1 \]

\[ \log(\log^{*} n) \]

\[ \frac{1}{n^{1/\log n}} = 2 \]
NOTES ON PROBLEM 2-3A

1. The relationship of double exponentiation and factorials.
   For all \( n > 2 \), \( n^n > n! \)
   Since \( n = 2^{\log n} \), \( n^n = \left(2^{\log n}\right)^n = 2^{n \log n} \)
   \[ \lim_{n \to \infty} \frac{2^n}{n^n} = \lim_{n \to \infty} \frac{2^n}{2^n \log n} = \lim_{n \to \infty} 2^{n - \log n} = \infty \]

2. The relationship of factorials and exponentials.
   It is well known that the radius of convergence for the following Taylor series is \((-\infty, \infty)\)
   \[ \sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x \]
   It is also known that for a series to be convergent, the sequence of terms must converge to zero. In other words, for all \( x \),
   \[ \lim_{n \to \infty} \frac{x^n}{n!} = 0 \]
   This implies that for any exponential function \( a^n, a^n \in o(n!) \)

3. The relationship of two exponentials.
   Let \( a \) and \( b \) be two positive real numbers, such that \( a > b \).
   \[ \lim_{n \to \infty} \frac{a^n}{b^n} = \lim_{n \to \infty} \frac{a^n}{a^{\log_a b} n} = \lim_{n \to \infty} a^{(1 - \log_a b) n} \]
   Since \( b < a \), \( \log_a b < \log_a a = 1 \), and \( 1 - \log_a b > 0 \).
   Therefore the function \( a^{(1 - \log_a b) n} \) is a strictly increasing exponential function and is unbounded. In other words,
   \[ \lim_{n \to \infty} a^{(1 - \log_a b) n} = \infty \], and \( a^n \in o(b^n) \)

4. The relationship of exponentials and polynomials.
   Let \( k \) be an integer, and \( a > 0 \) a real number.
   \[ \lim_{n \to \infty} \frac{a^n}{n^k} = \lim_{n \to \infty} \frac{(\ln a)a^n}{kn^{k-1}} = \lim_{n \to \infty} \frac{(\ln^2 a)a^n}{k(k-1)n^{k-2}} = \lim_{n \to \infty} \frac{(\ln a)a^n}{k!} = \infty \]

5. The relationship of two polynomials.
   Let \( \varepsilon > 0 \) and \( k > 0 \) be real numbers.
   \[ \lim_{n \to \infty} \frac{n^{k+\varepsilon}}{n^k} = \lim_{n \to \infty} n^\varepsilon = \infty \]

   Let \( a > 0 \), and \( k > 0 \) be real numbers.
   \[ \lim_{n \to \infty} \frac{n^k}{\log a + n} = \lim_{n \to \infty} \frac{\ln a}{\ln n} = \lim_{n \to \infty} \frac{(\ln a)k^{n-1}}{1 / n} = \lim_{n \to \infty} (\ln a)k^n = \infty \]

7. The relationship of polylogs and polynomials.
   Let \( a > 0 \) and \( m > 0 \) be real numbers, and let \( k > 0 \) be an integer.
\[
\lim_{n \to \infty} \frac{n^m}{\log_a n} = \lim_{n \to \infty} \frac{(\ln k a)(m)}{n^{m-1}} = \lim_{n \to \infty} \frac{k \ln k^{-1}}{n(1/n)} = \lim_{n \to \infty} \frac{(\ln k a)(m)}{n^{m-1}}
\]
\[
= \lim_{n \to \infty} \frac{(\ln k a)(m)}{k!} = \infty
\]

8. \((n + 1)! = n!(n + 1)\) Grows faster than \(n!\) and slower than \(2^{2^n}\).

9. \((\lg n)!\) grows faster than any polynomial, because of property 2 above. In property 2, substitute \(\lg n\) for \(n\), then we observe that \((\lg n)!\) grows faster than \(x^{\lg n}\) for any \(x > 0\). In particular, \((\lg n)!\) grows faster than \((2^k)^{\lg n} = n^k\). By property 1 (and the discussion) \(n^n\) grows faster than \(n!\), so \((\lg n)!\) grows faster than \((\lg n)!\).

10. Let \(f = (\lg n)^{\lg n}\). Then \(\lg f = \lg((\lg n)^{\lg n}) = \lg n \cdot \lg \lg n = \lg n^{\lg \lg n}\), and \(f = n^{\lg \lg n}\).

Note that \(n^{\lg n}\), where \(f(n)\) increases without bound, grows faster than \(n^k\) for any constant \(k\).

11. \(\lg(n!) = \lg(1 \cdot 2 \cdot 3 \cdot \ldots \cdot n) = \sum_{i=1}^{n} \lg n\)

The anti-derivative of \(\lg n\) is \((n \ln n - n) / \ln 2\). Using the integral formula:
\[
\int_a^{b+1} f(x)dx \leq \sum_{i=a}^{b} f(i) \leq \int_a^{b} f(x)dx 
\]
it is quite easy to obtain the \(\Theta(n \lg n)\) result.

12. The function \(2^{\sqrt{\lg n}} = n^{\frac{1}{\lg n}}\) because of the identity \(n^{1/\lg n} = 2\). Since \(\sqrt{\lg n}\) is a decreasing function \(n^{\frac{1}{\lg n}}\) grows slower than \(n^k\) where \(k\) is a constant. The relationship to the polylog functions can be obtained through differentiation. The function is differentiable in a straightforward (if somewhat complicated) way.

13. Note that \(\lg^* n\) grows slower than \(\lg n\), \(\lg \lg n\), \(\lg \lg \lg n\) or for any fixed number of iterations of the log function. Since \(\lg^* n\) grows slower than \(\lg \lg \lg n\), \(2^{\lg^* n}\) grows slower than \(2^{\lg \lg n} = \lg \lg n\). (Note that \(\lg \lg n = k \ln \ln n + j\) for constants \(k\) and \(j\).)

14. Since \(\lg^* n\) is the number of times that one must iterate the log to obtain a number less than 1, applying the log before computing \(\lg^*\) simply reduces the number of required iterations by 1.

15. Let \(f = n^{1/\lg n}\). Then \(\lg f = \lg(n^{1/\lg n}) = (\lg n)(1/\lg n) = 1\). Then \(f = 2^{\lg f} = 2^1 = 2\).