Proof That
Unique Edge Weights
Yield a Unique
Minimum Spanning Tree

by

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Theorem. If $G=(V,E)$ is a graph with unique edge weights, and if $G$ has two different spanning trees $S=(V,E_S)$ and $T=(V,E_T)$, then there is a third spanning tree $T'$ (possibly equal to $S$ or $T$) such that either $W(T') < W(S)$ or $W(T') < W(T)$.

Proof. Let $G=(V,E)$ be a graph with unique edge weights. Suppose $S=(V,E_S)$ and $T=(V,E_T)$ are two distinct spanning trees for $G$. Let $U_S=E_S-E_T$, and $U_T=E_T-E_S$. Since $S$ and $T$ are different, $E_S \neq E_T$. Since $|E_S|=|E_T|$, $U_S \neq \emptyset$, and $U_T \neq \emptyset$. Since $G$ has unique edge weights, there is an edge $e \in U_S \cup U_T$ of maximum weight. Without loss of generality, assume that $e \in U_T$. Let $T_x=(V,E_T-e)$. $T_x$ has two connected components $C_1$ and $C_2$. Since $S$ is connected, $S$ must contain an edge from a vertex $u$ in $C_1$ to a vertex $v$ in $C_2$. Furthermore, $(u,v)$ must be an element of $U_S$, otherwise $T$ would contain a cycle. Since $(u,v)$ is an element of $U_S$, the weight of $(u,v)$ must be less than the weight of $e$. Let $T'=(V,E_T-e \cup \{(u,v)\})$. $T'$ is a spanning tree of $G$, and $W(T') = W(T)-W(e)+W((u,v)) < W(T)$.

Corollary 1. If $G=(V,E)$ has unique edge weights, and $G$ has two different spanning trees $S=(V,E_S)$ and $T=(V,E_T)$, and $W(S)=W(T)$ then there is a third spanning tree $T'$ of $G$ not equal to either $S$ or $T$ such that $W(T') < W(T)$.

Proof: By the main theorem, $T'$ must exist and have a weight strictly smaller than one of $S$ and $T$. Since $S$ and $T$ are of the same weight, the weight of $T'$ must be less than both. Since the weight of $T'$ is less than the weight of $S$ or $T$, $T'$ cannot be equal to $S$ or $T$.

Corollary 2. If $G=(V,E)$ has unique edge weights then $G$ has a unique minimum spanning tree.

Proof. If there were two minimum spanning trees $S$ and $T$, and $S$ and $T$ were different, then by corollary 1 there must be a third minimum spanning tree $T'$ whose total weight is less than that of $S$ or $T$. Since the weight of $S$ and $T$ is minimal, this is impossible, therefore the minimum spanning tree of $G$ must be unique.