1 Let $S$ be a set of ordered pairs such that
Initial - $(0,0) \in S$
Step - $(a,b) \in S \rightarrow (a+2,b+3) \in S \land (a+3,b+2) \in S$

a) Show $S$ after applications of the step.
$S = \{(0,0), (2,3), (3,2), (4,6), (5,5), (6,4), (7,8), (8,7), (9,6), (8,12), (9,11), (10,10), (11,9), (12,8)\}$

b) Use strong induction on the number of applications of the step that $(a,b) \in S$.
$5 | (a+b)$, where $5 | (a+b)$ means $\frac{a+b}{5}$ has a remainder of 0.

Basis Step:
Initially $S=\{(0,0)\}$, $a=0$ and $b=0$. $\frac{a+b}{5}$ has a remainder of 0.
So $(a,b) \in S \rightarrow 5|(a+b)$

Inductive Hypothesis: Let the step is applied for $m$ number of times. $\forall (a,b) \in S \rightarrow 5|(a+b)$ for number of steps $\leq m$

Now $\forall (a,b) \in S \rightarrow 5|(a+b)$ is to be proved for $(m+1)^{th}$ application of the step.
In the $(m+1)^{th}$ application, the elements are added as follows:
Case 1: $(a+2,b+3)$
In the $m^{th}$ step all the $(a,b) \in S$ satisfies $5|(a+b)$. Now in the $(m+1)^{th}$ step the sum of elements of ordered pair is $a+2+b+3=(a+b+5)$ which is exactly divisible by 5. So, $\frac{a+b}{5}$ has a remainder of 0.

Case 2: $(a+3,b+2)$
In the $m^{th}$ step all the $(a,b) \in S$ satisfies $5|(a+b)$. Now in the $(m+1)^{th}$ step the sum of elements of ordered pair is $a+3+b+2=(a+b+5)$ which is exactly divisible by 5. So, $\frac{a+b}{5}$ has a remainder of 0.

So the given statement $(a,b) \in S \rightarrow 5|(a+b)$, where $5|(a+b)$ is proved.

2 Assume that a chocolate bar consists of $n$ squares arranged in a rectangular pattern. The entire bar, or any smaller rectangular piece of the bar, can be broken along a vertical or horizontal line separating the squares. Assuming that only one piece can be broken at a time, use strong induction to show that $n \leq 1$
breaks are needed to create n separate pieces.

Let \( H(n) \) be the number of breaks needed to create \( n \) separate pieces for a chocolate bar consisting of \( n \) squares. Basis Step:
For a chocolate bar of size 1, no breaks are required. \( H(1) = 1 - 1 = 0 \)
For a chocolate bar of size 2, no breaks are required. \( H(1) = 2 - 1 = 1 \)

Induction Step:
Let \( H(k) = k - 1 \) for all \( k \leq m \). So \( H(m) = m - 1, H(m-1) = m - 2 \) and so on.
Now we have to prove that \( H(m+1) = (m+1) - 1 = m \)
For a chocolate bar of size \( m+1 \), 1 break can be performed to produce 1 piece of with 1 square and another piece with \( (m-1) \) squares.
So \( H(m+1) = 1 + H(m) = 1 + (m-1) = m \)
\( H(m+1) = m \).
So \( m \) breaks are needed to break a bar of size \( m+1 \).
Hence it is proved that \( n - 1 \) breaks are needed to to create \( n \) separate pieces.

3 From the following functional definition, find a formula for \( f(n) \) and prove your formula correct.
\( f(0) = 0, f(1) = 1 \).
\( \forall n \geq 2, f(n) = 2 \cdot f(n - 1) \)
The formula for \( f(n) \) is :
\( f(n) = 0 \) if \( n = 0 \) else \( f(n) = 2^{n-1} \)

Proof:
Basis step:
for \( n = 0, f(0) = 0 \)
\( f(1) = 2^{1-1} = 2^0 = 1 \)
\( f(2) = 2^{2-1} = 2^1 = 2 \) which is equal to \( 2 \cdot f(1) \)

Induction step:
\( \forall k \leq m \) \( f(k) = 2^{k-1} = 2^k f(k-1) \)
We have to prove \( f(m+1) = 2^m \)
\( f(m+1) = 2^m f(m) \)
\( f(m+1) = 2^m 2^{m-1} = 2^m \).
Proved

4 Given \( A \) and \( A^n \) below. Prove that the matrix \( A \) raised to the \( n \)th power is \( A^n \), where \( f_n \) is the \( n \)th Fibonacci number.

Basis step:
For \( n = 1 \),
\( A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix} = A^1 \)
Proved for \( n = 1 \).
Induction step:
Let us assume that $A$ raised to the $k$th power is $A^k$ for all $k \leq m$.

So $A^m = \begin{bmatrix} f_{m+1} & f_m \\ f_m & f_{m-1} \end{bmatrix}$

We have to prove $A$ raised to $(m+1)$th power is $A^{m+1}$.

$A^{m+1} = A^* A^m$

$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} f_{m+1} & f_m \\ f_m & f_{m-1} \end{bmatrix}$

$= \begin{bmatrix} f_{m+1} + f_m & f_m + f_{m-1} \\ f_{m+1} + 0 & f_m + 0 \end{bmatrix} = \begin{bmatrix} f_{m+2} & f_{m+1} \\ f_{m+1} & f_m \end{bmatrix}$

Hence $A$ raised to the $n$th power is $A^n$.

5 Give a recursive algorithm with input $n$, an integer, for finding the sum of the squares of the first $n$ integers.

function sumSquared(n):
    if n==0 return 0
    else return n*n + sumSquared(n-1)
end