CSI 2350 Assignment 17

April 2019

For each of the following, if the recurrence relation is linear and homogeneous with constant coefficients, solve it by providing the closed form equation. If it is not, state why it is not.

1. \( \forall n \geq 2, a_n = a_{n-1} + 6a_{n-2}, a_0=3, a_1=6 \)
   - The characteristic equation of the recurrence relation is \( r^2-r-6=0 \). Its roots are \( r=3 \) and \( r=-2 \). Hence, the sequence \( \{a_n\} \) is a solution to the recurrence relation if and only if
     \[
     a_n = \alpha_1 3^n + \alpha_2 (-2)^n.
     \]
   From the initial conditions,
   \[
   a_0=3=\alpha_1 + \alpha_2 \\
   a_1=6=\alpha_1 3 + \alpha_2 (-2)
   \]
   Solving this we get \( \alpha_1=12/5 \) and \( \alpha_2=3/5 \)
   Hence the solution to the recurrence relation is the solution \( \{a_n\} \) with
   \[
   a_n = \frac{12}{5} 3^n + \frac{3}{5} (-2)^n.
   \]

2. \( \forall n \geq 2, a_n = 7a_{n-1} - 10a_{n-2}, a_0=2, a_1=1 \)
   - The characteristic equation of the recurrence relation is \( r^2-7r+10=0 \). Its roots are \( r=5 \) and \( r=2 \). Hence, the sequence \( \{a_n\} \) is a solution to the recurrence relation if and only if
     \[
     a_n = \alpha_1 5^n + \alpha_2 2^n.
     \]
   From the initial conditions,
   \[
   a_0=2=\alpha_1 + \alpha_2 \\
   a_1=1=\alpha_1 5 + \alpha_2 2
   \]
   Solving this we get \( \alpha_1=-1 \) and \( \alpha_2=3 \)
   Hence the solution to the recurrence relation is the solution \( \{a_n\} \) with
   \[
   a_n = (-1) 5^n + 3 2^n.
   \]

3. \( \forall n \geq 2, a_n = 2a_{n-1} +1, a_0=5 \)
   - \( a_n = 2a_{n-1} +1 \). The recurrence relation contains 1 which is not in terms of previous members of the relation. So it is not linear and homogenous with
4. \( \forall n \geq 2, a_n = 6a_{n-1} - 8a_{n-2}, a_0=4, a_1=10 \)
- The characteristic equation of the recurrence relation is \( r^2 - 6r + 8 = 0 \). Its roots are \( r=4 \) and \( r=2 \). Hence, the sequence \( \{a_n\} \) is a solution to the recurrence relation if and only if
\[
a_n = \alpha_1 4^n + \alpha_2 2^n.
\]
From the initial conditions,
\[
a_0=4=\alpha_1 + \alpha_2 \\
a_1=10=4\alpha_1 + 2\alpha_2
\]
Solving this we get \( \alpha_1=1 \) and \( \alpha_2=3 \)

Hence the solution to the recurrence relation is the solution \( \{a_n\} \) with
\[
a_n = 4^n + 3 \cdot 2^n.
\]

5. \( \forall n \geq 2, a_n = 2a_{n-1} - a_{n-2}, a_0=4, a_1=1 \)
- The characteristic equation of the recurrence relation is \( r^2 - 2r + 1 = 0 \). Its root is \( r=1 \). Hence, the sequence \( \{a_n\} \) is a solution to the recurrence relation if and only if
\[
a_n = \alpha_1 1^n + \alpha_2 n 1^n.
\]
From the initial conditions,
\[
a_0=4=\alpha_1 + \alpha_2 \times 0 \text{ so } \alpha_1=4 \\
a_1=1=\alpha_1 + \alpha_2 \text{ so } \alpha_2 = -3
\]
Hence the solution to the recurrence relation is the solution \( \{a_n\} \) with
\[
a_n = 4 - 3 \cdot n
\]

6. \( \forall n \geq 2, a_n=a_{n-2} - a_{n-1}, a_0=5, a_1=-1 \)
- The characteristic equation of the recurrence relation is \( r^2 - 1 = 0 \). Its roots are \( r=1 \) and \( r=-1 \). Hence, the sequence \( \{a_n\} \) is a solution to the recurrence relation if and only if
\[
a_n = \alpha_1 1^n + \alpha_2 (-1)^n.
\]
From the initial conditions,
\[
a_0=5=\alpha_1 + \alpha_2 \\
a_1=-1=\alpha_1 - \alpha_2
\]
Solving this we get \( \alpha_1=2 \) and \( \alpha_2=3 \)

Hence the solution to the recurrence relation is the solution \( \{a_n\} \) with
\[
a_n = 2 + 3 \cdot (-1)^n
\]

7. \( \forall n \geq 2, a_n = -6a_{n-1} - 9a_{n-2}, a_0=3, a_1=-3 \)
- The characteristic equation of the recurrence relation is \( r^2 + 6r + 9 = 0 \). Its root is \( r=-3 \). Hence, the sequence \( \{a_n\} \) is a solution to the recurrence relation if and only if
\[ a_n = \alpha_1(-3)^n + \alpha_2n(-3)^n. \]

From the initial conditions,
\[ a_0 = 3 = \alpha_1 \]
\[ a_1 = -3 = -3\alpha_1 + \alpha_2(-3) \text{ so } \alpha_2 = -2 \]

Hence the solution to the recurrence relation is the solution \( \{a_n\} \) with
\[ a_n = 3(-3)^n - 2n(-3)^n \]

8. \( \forall n \geq 2, a_n = -4a_{n-1} + 5a_{n-2}, a_0 = 2, a_1 = 8 \)
- The characteristic equation of the recurrence relation is \( r^2 + 4r - 5 = 0 \). Its roots are \( r = -5 \) and \( r = 1 \). Hence, the sequence \( \{a_n\} \) is a solution to the recurrence relation if and only if
\[ a_n = \alpha_1(-5)^n + \alpha_21^n. \]
From the initial conditions,
\[ a_0 = 2 = \alpha_1 + \alpha_2 \]
\[ a_1 = 8 = \alpha_1(-5) + \alpha_2 \]
Solving this we get \( \alpha_1 = -1 \) and \( \alpha_2 = 3 \)

Hence the solution to the recurrence relation is the solution \( \{a_n\} \) with
\[ a_n = 3(-5)^n + 3 \]

9. \( \forall n \geq 1, a_n = n \cdot a_{n-1}, a_0 = 1 \)
\[ a_n = n \cdot a_{n-1}. \] The recurrence relation contains coefficient \( n \) before \( a_{n-1} \), which is not constant. So it is not linear and homogenous with constant coefficients.

10. \( \forall n \geq 1, a_n = 4a_{n-1}, a_0 = 2 \)
- The characteristic equation of the recurrence relation is \( r - 4 = 0 \). The root of the equation is \( r = 4 \).
Hence, the sequence \( \{a_n\} \) is a solution to the recurrence relation if and only if
\[ a_n = \alpha_1(4)^n \]
Using initial condition \( a_0 = 2 \), we get \( \alpha_1 = 2 \)
So the solution to the recurrence relation is the solution \( \{a_n\} \) with
\[ a_n = 2(4)^n \]