Section 1

Binary, Octal and Hex
Overview

- Needs to be added for Spring 2019
Section 2

Propositional Logic
Propositional Logic

- Needs to be added for Spring 2019
Section 3

Knights and Knave
Knights and Knaves

- Needs to be added for Spring 2019
Section 4

CNF, DNF and Proofs
CNF, DNF and Proofs

- Needs to be added for Spring 2019
Section 5

Set Theory
Sets

- Needs to be added for Spring 2019
Section 6

Functions and Sequences
Functions and Sequences

- Needs to be added for Spring 2019
Section 7

Summations and Matrices
Summations

- Add the terms in a sequence
- Uses $\Sigma$
- Examples:
  - $\sum_{j=1}^{n} j = \frac{n(n + 1)}{2}$
  - $\sum_{j=1}^{n} j^2 = \frac{n(n + 1)(2n + 1)}{6}$
  - $\sum_{j=1}^{n} 2j + 3 = 2\sum_{j=1}^{n} j + \sum_{j=1}^{n} 3 = 2\frac{n(n + 1)}{2} + 3n = n^2 + 4n$
  - $\sum_{j=0}^{n} ar^j = \frac{ar^{n+1} - a}{r - 1}[r \neq 1], (n + 1)a[r = 1]$
  - Let $S = \{1, 2, 4, 8\}$ then $\sum_{x\in S} x = 1 + 2 + 4 + 8 = 15$
Double Summations

- Summation over two variables
- Inner and outer loop
- Examples:
  - $\sum_{i=1}^{4} \sum_{j=1}^{3} i \times j = 1 + 2 + 3 + 2 + 4 + 6 + 3 + 6 + 9 + 4 + 8 + 12 = 60$
  - $\sum_{i=1}^{4} \sum_{j=1}^{3} i \times j = \sum_{i=1}^{4} (i + 2i + 3i) = \sum_{i=1}^{4} 6i = 6 \times \sum_{i=1}^{4} = 6 \times 10 = 60$
Countable and Uncountable Sets

- Sets $A$ and $B$ have the same cardinality iff there is a 1-1 correspondence (1-1 and onto function) from $A$ to $B$
- Finite sets are countable
- Infinite sets with 1-1 correspondence to integers are countable
- Set of real numbers is uncountable
- Fun fact: Some problems cannot be solved by a computer. Halting problem (does program P halt?).
Matrices

- Express relationships between elements of sets
- Example - Powers of numbers (Matrix M)

\[ M = \begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 1 & 2 & 4 \\
3 & 1 & 3 & 9 \\
\end{bmatrix} \]

- m rows and n cols is m x n matrix (M is 3x4).
- m=n is square matrix
- Element \( a_{i,j} \) where i is row number and j is column number
- \( M_{2,2} = 1 \), \( M_{3,3} = 3 \)
Addition

- Requires $M, N$ be same size
- $M + N = L \Rightarrow \forall i, j L_{i,j} = M_{i,j} + N_{i,j}$

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 1 & 2 & 4 \\
3 & 1 & 3 & 9 \\
\end{bmatrix}
+ \begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
\end{bmatrix}
= \begin{bmatrix}
2 & 3 & 4 & 5 \\
7 & 7 & 9 & 12 \\
12 & 11 & 14 & 21 \\
\end{bmatrix}
\]
Multiplication

- Columns in M must equal rows in N
- M is $m \times k$, N is $k \times n$, L is $m \times N$
- $L_{i,j} = \sum_{h=1}^{k} M_{i,h} \times N_{h,j}$

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 1 & 2 & 4 \\
3 & 1 & 3 & 9 \\
\end{bmatrix}
\times
\begin{bmatrix}
1 & 2 \\
5 & 6 \\
9 & 10 \\
11 & 12 \\
\end{bmatrix}
= 
\begin{bmatrix}
26 & 30 \\
69 & 78 \\
134 & 150 \\
\end{bmatrix}
\]

- Example:
  $L_{2,2} = \sum_{h=1}^{4} M_{2,h} \times N_{h,2} = 2 \times 2 + 1 \times 6 + 2 \times 10 + 4 \times 12 = 78$

- Example:
  $L_{1,2} = \sum_{h=1}^{4} M_{1,h} \times N_{h,2} = 1 \times 2 + 1 \times 6 + 1 \times 10 + 1 \times 12 = 30$

- NOT commutative (e.g., $M \times N \neq N \times M$, see above)
Identity Matrix and Transposition

- **Identity matrix (usually $I$)**
  - Square
  - Diagonal values are 1; all other values are 0
  - $M \times I = M$

- **Transpose of $M = M^t$**
  - $M^t_{i,j} = M_{j,i}$
  - $M$ is symmetric if $M = M^t$ (note: must be square)

\[
M = \begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 1 & 2 & 4 \\
3 & 1 & 3 & 9 \\
\end{bmatrix} \quad \rightarrow \quad M^t = \begin{bmatrix}
1 & 2 & 3 \\
1 & 1 & 1 \\
1 & 2 & 3 \\
1 & 4 & 9 \\
\end{bmatrix}
\]
Section 8

The Big $\Omega\Theta$ Fraternity
Binary Search

- Input list of elements $[a_1, a_2, \ldots, a_n]$, search key $x$
- Output index of element matching $x$, or 0 if not found
- Pseudocode
  - $i \leftarrow 1$
  - $j \leftarrow n$
  - while $(i < j)$
    - $m \leftarrow \lfloor (i + j)/2 \rfloor$
    - if $x > a_m$ then $i \leftarrow m + 1$
    - else $j \leftarrow m$
  - if $x = a_i$ then return $i$
  - else return 0
Binary Search Runtime

- Assume $n = 2^k$ elements in list
- After iteration, remaining list half of previous list size
- After $k$ iterations, list is size 1
- $\log(n) = k$
- Worst case running time is $\log(n)$
Let $f$ and $g$ be functions. $f(x)$ is $O(g(x))$ if $\exists C, k$ such that $f(x) \leq C \cdot g(x)$ for all $n > k$. Assume $x \geq 0$.

Function $f$ grows slower than function $g$ for $n > k$.

NOTE: Can say ”is” or $\in$ but $=$ is misleading (although used frequently).

NOTE: Existence proof. Find one $C$ and one $k$ (many may exist).

Technique: Start with known ($x > k$). Remember, we can select $k$.

Apply formulas to generate $C \cdot g(x) \geq f(x)$.

Examples:
- $f(x) = 17x + 11 \in O(g(x) = x^2)$. Let $C = 2$, $k = 17$.
  - $x \geq 17 \rightarrow x^2 \geq 17x$
  - $\rightarrow x^2 + x^2 \geq 17x + 11$
  - $\rightarrow 2 \cdot x^2 \geq 17x + 11$
Big O

- **Examples:**
  - \( f(x) = x \log(x) \in O(g(x) = x^2) \). Let \( C = 1, k = 1 \)
    - \( x \geq 1 \rightarrow x \geq \log(x) \rightarrow x^2 \geq x \log(x) \)
  - \( f(x) = x^2 + x + 1 \in O(g(x) = x^2) \) Let \( C = 3, k = 1 \)
    - \( x \geq 1 \rightarrow x^2 \geq x \rightarrow x^2 + x^2 \geq x^2 + x \)
    - \( \rightarrow x^2 + x^2 + x^2 \geq x^2 + x + 1 \rightarrow 3 \times x^2 \geq x^2 + x + 1 \)

- To show \( f(x) \) is not \( O(g(x)) \) must show no such \( C \) and \( k \) can exist.

- **Example:**
  - \( f(x) = x^2 \) is not \( O(g(x) = x) \).
  - Assume not. Therefore \( \exists C, k \) such that \( f(x) \leq C \times g(x) \forall x > k \)
  - Therefore, \( x^2 \leq C \times x \forall x \geq k \).
  - Therefore, \( x \leq C \forall x \geq k \), which is a contradiction.
Let $f$ and $g$ be functions. $f(x)$ is $\Omega(g(x))$ if $\exists C, k$ such that $f(x) \geq C \times g(x)$ for all $n > k$ Assume $x \geq 0$.

Function $f$ grows faster than function $g$ for $n > k$.

$f(x)$ in $\Omega(g(x)) \iff g(x) \in O(f(x))$.

Example:

- $f(x) = x^2$ is $\Omega(g(x) = x)$ because $g(x) = x$ is $O(f(x) = x^2)$
- $f(x) = x^4/2$ is $\Omega(g(x) = x^2)$. Show $g(x) = x^2$ is $O(f(x) = x^4/2)$
- Let $C = 1, k = 2$
- $x \geq 2 \rightarrow x^2 \geq 2x \rightarrow x^3 \geq 2x^2 \rightarrow x^4 \geq 2x^2 \rightarrow x^4/2 \geq x^2$
Let $f$ and $g$ be functions. If $f(x)$ is $\Omega(g(x))$ and $f(x)$ is $O(g(x))$, then $f(x)$ is $\Theta(g(x))$.

Equivalent: $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$.

$f(x)$ is bounded above and below by $g(x)$.

Example:
- $f(x) = x^2 + x + 1$ is $\Theta(g(x) = x^2)$
- Show $f(x)$ is $O(g(x))$, previous slide.
- Show $g(x)$ is $O(f(x))$. Let $C = 1, k = 1$.
  
  $x \geq 1 \rightarrow x^2 + x \geq x^2 \rightarrow x^2 + x + 1 \geq x^2$. 
Intractable Problems

- Show $f(x) = x^2$ is $O(g(x) = 2^x)$ Let $C = 1, k = 4$

- $x \geq 4 \rightarrow x \geq 2 \log(x) \rightarrow x \geq \log(x^2) \rightarrow \log(2^x) \geq \log(x^2) \rightarrow 2^x \geq x^2$

- Show $g(x) = 2^x$ is NOT $O(f(x) = x^2)$. Assume not.

- $\exists C, k$ such that $Cx^2 \geq 2^x, \forall x \geq k$.

- Therefore, $\log(Cx^2) \geq \log(2^x), \forall x \geq k$

- Let $a = \log(C)$. Therefore, $a + 2 \log(x) \geq x, \forall x \geq k$

- Note that $\forall a \exists x$ s.t. $x > a + 2 \log(x)$. Consider $\max(x = 2^a, 16)$. Then $2^a > 3a$, which is true for $a \geq 4$.

- Therefore, problems requiring exponential time are “harder” than quadratic problems (or any polynomial)

- Such problems are called **intractable**
P=NP?

- P - class of problems solvable in polynomial time
- NP - class of problems best solutions require exponential time
- Unknown if NP problems can be solved in polynomial time
- One of grand challenges of mathematics for 21st century (Millennium Problems)
- Example: Satisfiability
- Predicate $P$ in CNF $(p \lor q \lor r) \land (\neg p \lor s \lor \neg t) \ldots$
- Can assign truth values to variables such that $P$ can be satisfied?
- DNF is trivial $(p \land q \land r) \lor (\neg p \land s \land \neg t) \ldots$
Section 9

Induction
Proof technique for showing infinite series is true
Require incremental progress
Basis – Initial step (usually very easy to show)
Induction Hypothesis (IH) – Theorem is true for $k$ elements
NOTE! Have not proved it is true. Just assuming it is.
Show Theorem is true for $k + 1$ elements
Therefore, initially true. Basis is now IH for next step. Now have IH for following step, etc.
Induction Example

- Summations “easily” shown by induction (proof simple; algebra can be tricky)
- Prove $\sum_{i=1}^{n} i^3 = (n(n+1)/2)^2$
- Basis: $P(1)$. $1^3 = 1 = (1(2)/2)^2$
- IH: $\sum_{i=1}^{k} i^3 = (k(k + 1)/2)^2$
- Let $n = k + 1$.
- $\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k + 1)^3$
- By IH, $\sum_{i=1}^{k+1} i^3 = (k(k + 1)/2)^2 + (k + 1)^3$
- $= (((k^2 + k)/2)^2 + k^3 + 3k^2 + 3k + 1$
- $= (k^4 + 2k^3 + k^2)/4 + k^3 + 3k^2 + 3k + 1$
- $= (k^4 + 2k^3 + k^2 + 4k^3 + 12k^2 + 12k + 4)/4$
- $= (k^4 + 6k^3 + 13k^2 + 12k + 4)/4$
- Note that $((k + 1)(k + 2)/2)^2 = ((k^2 + 3k + 2)/2)^2 = (k^4 + 3k^3 + 2k^2 + 3k^3 + 9k^2 + 6k + 2k^2 + 6k + 4)/4$
- $= (k^4 + 6k^3 + 13k^2 + 12k + 4)/4$
- So $\sum_{i=1}^{k+1} i^3 = ((k + 1)(k + 2)/2)^2$
Induction Example

- Prove $x \geq 4 \rightarrow x \geq 2 \times \log(x)$
- Basis: $P(4)$. $4 \geq 2 \times \log(4) = 4$
- IH: $x = k \rightarrow x \geq 2 \times \log(x)$
- Let $x = k + 1$.
- By IH, $x \geq 2 \times \log(k) + 1$
- $x \geq \log(k^2) + \log(2)$
- $x \geq \log(2 \times k^2)$
- $x > \log(k^2 + 2 \times k + 1)$
- $x > \log((k + 1)^2) = 2 \times \log(k + 1) = 2 \times \log(x)$
Tag! You’re Out!

Need four volunteers.
Odd Man Pie Fights (from text)
- People stand in yard at distinct distances
- Throw pie at nearest neighbor
- Odd number of participants ensures at least 1 person is not hit

Technique - Induct over $n$ for $P(2n + 1)$

Basis: $n = 1$, $P(3)$. Let $(a, b)$ be closest pair. Then $a$ hits $b$ and $b$ hits $a$. $c$ hits whoever is closest to $c$, but nobody hits $c$.
Odd Man Out Proof (cont’d)

- IH: Assume theorem is true for $n = k$, $P(2 \times k + 1)$.
- Let $n = k + 1$. Therefore, want to show $P(2 \times (k + 1) + 1)$ or $P(2 \times k + 3)$
- Let $(a, b)$ be the closest pair (all distances unique implies least element exists)
- Therefore, $a$ hits $b$ and $b$ hits $a$.
- Case I:
  - Someone else throws a pie at $a$ or $b$
  - Therefore, at least 3 pies thrown at $a$ and $b$, leaving at most $2k$ pies for $2k + 1$ people
- Case II:
  - Nobody throws a pie at $a$ or $b$
  - Now $2 \times (k + 1)$ people remain with pies
  - By IH, at least one is not hit
Induction Example

- Number of elements in $2^S$ (from text)
- Show $|2^S| = 2^{|S|}$
- Basis: $\emptyset$. The only subset of $\emptyset$ is itself. $|2^\emptyset| = 1 = 2^0$
- IH: $|S| = k \implies |2^S| = 2^k$
- Let $|T| = k + 1$. $T = S \cup \{a\}$ such that $S = T - \{a\}$.
- Let $X \subseteq S$. Therefore, $X \subseteq T$ and $X \cup \{a\} \subseteq T$.
- By IH, there are $2^k$ such subsets of $S$, and $2 \times 2^k = 2^{k+1}$ subsets of $T$. 
Section 10

Complete Induction & Recursion
Strong (Complete) Induction

- IH is not just that $P(k)$ is true, but $P(1) \land P(2) \land P(3) \ldots P(k)$ is true
- Equivalent to Induction, but sometimes easier to use
- Example: Prove that every positive integer $n$ can written as a sum of distinct powers of 2
  - Basis: $n = 1 \Rightarrow 2^0 = 1 = n$
  - IH: $\forall n \leq k, n$ can be written as a sum of distinct powers of 2
  - Let $n = k + 1$
  - Case I:
    - Let $n$ be odd.
    - Since $n$ is odd, $k$ must be even.
    - By IH, let $S$ be representation of the sum for $k$
    - Therefore, $2^0$ cannot be in $S$
    - Therefore $n = (\text{representation for } k) + 2^0$. 
Proof Continued

- Case II
  - Let $n$ be even.
  - Therefore, $n/2$ is an integer such that $n/2 \leq k$.
  - By IH, let $S$ be representation of the sum for $n/2$.
  - Multiplying $S$ by 2 increases each exponent by 1.
  - Therefore, $S$ with each exponent increased by 1 is $n$. 

Function Recursion

- Compute values of a function based on previous values in the function
- Specify the value at 0 (or first \(k\) values)
- Provide rule for \(f(n)\) based on lower values of \(n\)
- Factorial: \(f(0) = 1. \forall n, n > 0 \rightarrow f(n) = n \times f(n - 1)\).
- Fibonacci:
  \[f(0) = 0, f(1) = 1. \forall n, n > 1 \rightarrow f(n) = f(n - 1) + f(n - 2)\]
- Paradox:
  \[f(0) = 0.5. \forall n, n > 1 \rightarrow f(n) = (1 - f(n - 1))/2 + f(n - 1)\]
- Min: \(f(1) = a_1. \forall n, n > 1 \rightarrow f(n) = \min(a_n, f(n - 1))\)
Set Recursion

- Compute elements in a set based on previous elements in the set.
- Provide rule for initial elements. Provide rule for adding new elements.
- Initial - $1 \in S$. Step - $s, t \in S \rightarrow s + t \in S$ Prove $S = \mathbb{Z}^+$
  (Note! $s$ and $t$ do not have to be unique!):
    - Clearly, $S \subseteq \mathbb{Z}^+$
    - Basis: $1 \in S$ by definition of $S$
    - IH: $k \in S$.
      - Since $k$ and $1$ are in $S$, by rule, $k + 1 \in S$
- Transitive Closure: Let $V = \{v_0 \ldots v_{n-1}\}$ and $E \subset V \times V$. Initial - $(v_i, v_j) \in E \rightarrow (v_i, v_j) \in S$. Step -
  $(v_i, v_j), (v_j, v_k) \in S \rightarrow (v_i, v_k) \in S$
- DAG on the board
Recursive Algorithm

- Compute results using previously computed results
- Provide rule for terminal case. Provide rule for recursive call.
- Example: factorial
  - fact(n)
  - if $n \leq 0$ return 1
  - else return $n \times \text{fact}(n-1)$
- Example: gcd
  - gcd(a,b)
  - if $a=0$ return b
  - return gcd(b%a,a)
Recursive Algorithm

Example: fibonacci (Problem?)

- \text{fib}(a)
- \text{if } a=0 \text{ return } 0
- \text{if } a=1 \text{ return } 1
- \text{return } \text{fib}(a-1)+\text{fib}(a-2)

Example: Binary Search (Problem?)

- Let \( A = [a_1, a_2, \ldots, a_n] \)
- \text{find}(A, x, i, j)
- \text{if } i \geq j, \text{ then}
  - \text{if } a_i = x \text{ return } i
  - \text{else return } 0
- \( m \leftarrow \lfloor (i + j)/2 \rfloor \)
- \text{if } x > a_m, \text{ then return } \text{find}(A, x, m + 1, j)
- \text{else return } \text{find}(A, x, i, m)
Loop Invariants

- Only part of Chapter 5 we’ll cover
- Proposition true for every iteration of loop
- Example:
  - Loop invariant: \( f = i! \)
  - \( i \leftarrow 1 \)
  - \( f \leftarrow 1 \)
  - while \((i < n)\)
    - \( i \leftarrow i + 1 \)
    - \( f \leftarrow f \times i \)
  - At end of loop, loop invariant and \( \neg (i < n) \rightarrow f = n! \)
Section 11

Basic Counting
Overview

- Needs to be added for Spring 2019
Section 12

Permutations and Combinations
Overview

- Permutations are the number of arrangements of a set of items of a specific size
- Combinations are the number of subsets of a specific size
- The Binomial Theorem is a general representation of binomial coefficients
- Playing cards
  - 4 suits - spades, hearts, diamonds, clubs
  - 13 values - A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2
  - 52 total cards (+ jokers)
Permutations

- Battle - Card game where highest value wins (special rules for ties)
- Demonstration (Hearts only)
  - How many ways to arrange 6 cards? (Hint: Product rule.)
  - How many ways to arrange 3 of 6 cards?
  - How many ways to arrange k of 6 cards?
  - How many ways to arrange k of n cards?
Permutation Formulae

- All arrangements: \( n(n - 1)(n - 2) \ldots (1) = n! \)
- Arrangements of size \( r \): \( n(n - 1)(n - 2) \ldots n - (r - 1) \)
  - Last term is also \( n - r + 1 \)
  - \( (n - r)! = (n - r)(n - r - 1)(n - r - 2) \ldots (1) \)
  - \( \frac{n!}{(n-r)!} \) yields arrangements of size \( r \)
- Formula: \( P(n, r) = \frac{n!}{(n-r)!} \)
- Note: if \( n = r \) then \( P(n, n) \) or \( P(n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! \)
Combinations

- Oh Heck - Card game with hands and “tricks”
- Demonstration (Hearts only)
  - Order of cards in hand does not matter
  - How many different hands of size 3 can be dealt?
  - Given my hand, how many different hands of size 3 can opponent have?
  - Given my hand, how many different hands of size 3 can opponent have with all cards lower than my highest?
Combination Formulae

- Number of permutations divided by the number that are the same (division rule)
  \[ P(n, r)/P(r) = \frac{n!}{(n-r)!} = \frac{n!}{r!(n-r)!} \]

- Number of subsets of set \( S \) a given size
  - Size 0 is \( \emptyset \), only one \( \frac{n!}{0!(n-0)!} = 1 \)
  - Size \( n \) is \( S \), only one \( \frac{n!}{n!(n-n)!} = 1 \)
  - Size 1 subsets, each element is \(|S|\), \( \frac{n!}{1!(n-1)!} = n \)
  - Size \( n-1 \) subsets, \( S \) with each element removed, number is \(|S|\), \( \frac{n!}{(n-1)!(n-(n-1))!} = n \)

- Formula: \( C(n, r) \) or \( \binom{n}{r} = \frac{n!}{r!(n-r)!} \)

- Note: \( \binom{n}{n} = \frac{n!}{n!0!} = 1 \) and \( \binom{n}{0} = \frac{n!}{n!0!} = 1 \)
Binomial Theorem

\[ (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \]
\[ (0)x^n + (1)x^{n-1}y + (2)x^{n-2}y^2 + \ldots (n-1)xy^{n-1} + (n)y^n \]

- Can find coefficient for any term in binomial expansion
  - Given \((2x + 3y)^4\), what is the coefficient for the \(x^2y^2\) term?
    - \( \binom{4}{2}(2x)^2(3y)^2 = 6 \times 4 \times 9 = 216 \)
  - Shows \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \) (from text)
  - \( 2^n = (1 + 1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k}1^k = \sum_{k=0}^{n} \binom{n}{k} \)
Pascal’s Triangle

- Example on board
- Pascal’s Identity: \( \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \)
- Algebraic proof:
  - \( \binom{n+1}{k} = \frac{(n+1)!}{k!(n+1-k)!} \)
  - \( \binom{n}{k-1} = \frac{n!}{(k-1)!(n-(k-1))!} \)
  - \( = \frac{k*(k-1)!*(n+1-k)!}{n!} \)
  - \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \)
  - \( = \frac{(n+1-k)n!}{k!(n-k)!(n+1-k)!} = \frac{(n+1-k)n!}{k!(n+1-k)!} \)
  - \( \binom{n}{k-1} + \binom{n}{k} = \frac{k*n!}{k*(k-1)!(n+1-k)!} + \frac{(n+1-k)n!}{k!(n+1-k)!} \)
  - \( = \frac{k*n!+(n+1-k)n!}{k!(n+1-k)!} = \frac{n*(k+n-1)}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!} \)
Section 13

Probability
Finite Probability

- Let $S$ be a set of equally likely outcomes – sample space.
- Let $E \subseteq S$ be a set of desired outcomes – event.
- $p(E) = \frac{|E|}{|S|}$ – probability of $E$

Examples (sample space is deck of cards):
- Probability of drawing a heart $p(\text{heart}) = \frac{13}{52} = \frac{1}{4}$
- Probability of drawing a king $p(\text{king}) = \frac{4}{52} = \frac{1}{13}$
- Probability of drawing king of hearts $p(\text{Kheart}) = \frac{1}{52}$
- Probability of drawing two hearts in a row (replacing your card) $p(\text{2hearts}) = \frac{169}{2704}$
  - Why? Two draws makes total outcomes is $52 \times 52$
  - 13 successes in each draw means 169 successful outcomes (product rule)
- Probability of drawing 5 hearts in a row (without replacement)
  $p(\text{flush}) = \frac{13 \times 12 \times 11 \times 10 \times 9}{52 \times 51 \times 50 \times 49 \times 48}$
Complements and Unions

- The complement of \( E \) is \( \bar{E} \) is \( S - E \). \( p(\bar{E}) = 1 - p(E) \)
- Probability of not drawing a heart \( p(\text{heart}) = 1 - 1/4 = 3/4 \)
- Sometimes much easier to calculate the complement
- The union of two events is \( E_1 \cup E_2 \).
  \[ p(E_1 \cup E_2) = p(E_1) + P(E_2) - P(E_1 \cap E_2) \]
- \( p(\text{kingorheart}) = p(\text{king}) + p(\text{heart}) - p(K\text{heart}) = 1/13 + 1/4 - 1/52 = 16/52 \)
Let $S$ be a sample space

Each element in $S$ is assigned a probability $(p(s))$.

$0 \leq p(s) \leq 1$

$\sum_{s \in S} p(s) = 1$

Function from $S$ to set of probabilities is called probability distribution function

If all probabilities are the same, uniform distribution $|S| = n, \forall s \in S, p(s) = 1/n$

$p(E) = \sum_{s \in E} p(s)$
Conditional Probability

- Probability of event $E$ given that event $F$ has happened $\Pr(E|F)$

\[
\Pr(E|F) = \frac{\Pr(E \cap F)}{\Pr(F)}
\]

- Example:
  - Given that 4 hearts in a row have been drawn, what is the probability that a 5th heart will be drawn?
  - $\Pr(F) = \Pr(4\text{ hearts}) = \frac{13 \times 12 \times 11 \times 10}{52 \times 51 \times 50 \times 49}$
  - In this case, $\Pr(E \cap F) = \Pr(E) = \Pr(\text{flush}) = \frac{13 \times 12 \times 11 \times 10 \times 9}{52 \times 51 \times 50 \times 49 \times 48}$
  - $\Pr(E|F) = \frac{9}{48}$
Independence

- Two events are independent if one happening has no effect on the other happening.
- Examples:
  - I draw a 7 from a deck of cards and you draw a 10 from a different deck.
  - I wear a hat and President Livingstone wears a hat.
  - We have an exam in 2350 and Dr. Donahoo gives an exam in 4321.
- $E$ and $F$ are independent iff $p(E) \times p(F) = p(E \cap F)$
- Different decks:
  - $p(7) = 1/13, p(10) = 1/13, p(7 \land 10) = 1/169 \approx .0060$
- Same decks: $p(7) = 1/13, p(10) = 1/13, p(7 \land 10) = 1/13 \times 4/51 = 4/663 \approx .0059$
Bernoulli Trials

- Probability of $k$ successes of $n$ independent trials with success $p$ and failure $q = 1 - p$ is $\binom{n}{k} p^k q^{n-k}$

- Examples (replacing cards back in deck)
  - Probability of drawing 4 hearts out of 5 cards
    - $n = 5$, $k = 4$, $p = .25$, $q = .75$, $p(4H) = \binom{5}{4}.25^4.75 \approx .0146$
  - Probability of drawing at least 4 hearts out of 5 cards
    - $p(4H) + p(5H) = \binom{5}{4}.25^4.75 + \binom{5}{5}.25^5.75^0 = 0.015625$
  - Probability of drawing at least 2 hearts out of 5 cards
    - $1 - (p(0H) + p(1H)) = 1 - \binom{5}{0}.25^0.75^5 + \binom{5}{1}.25^1.75^4 \approx 0.367$
Random Variable

- Not random and not a variable
- Function from $S$ to $\mathbb{R}$ (assigns real number to each possible outcome)
- Example: (Replacing cards as before)
  - Draw 3 cards from a deck. The set of all possible outcomes is $S$.
  - Let $X(s)$ be the random variable of the number of times a heart is drawn, where $s \in S$
  - Quick demo
  - $\forall s \in S, 0 \leq X(s) \leq 3$
- Distribution of $X$ on $S$ is the set of pairs $(r, p(X = r))$ where $r \in X(s)$
- From example:
  - $(0, 0.421875), (1, 0.421875), (2, 0.140625), (3, 0.015625)$
Section 14

Expected Value and Variance
Expected Value

- Given a random variable $X$, the *expected value* of $X$ is
  $$E(X) = \sum_{s \in S} p(s)X(s)$$
- Recall $X(s) = y$ means $y$ is the number of interesting occurrences in event $s$
- Examples:
  - Assume cards 2-10 of hearts. Let $X$ be the value of the card. Expected value of drawing a card:
    $$1/9 \times 2 + 1/9 \times 3 + 1/9 \times 4 + 1/9 \times 5 + 1/9 \times 6 + 1/9 \times 7 + 1/9 \times 8 + 1/9 \times 9 + 1/9 \times 10 = 6$$
  - Assume J,Q,K have value 10 and A has value 11. Expected value of a card:
    $$E(X) = 1/13 \times (\sum_{k=2}^{9} k + 11) + 4/13 \times 10 = 95/13$$
  - Assume 5 cards, consisting of 4 2s and 1 3.
    $$E(X) = .8 \times 2 + .2 \times 3 = 2.2$$
Bernoulli Expected Value

The expected number of success of \( n \) Bernoulli trials with success \( p \) is \( n \cdot p \).

Proof: (from text)

Let \( X(s) \) be the number of successes out of \( n \) trials.

\[ p(X = k) = \binom{n}{k} p^k q^{n-k} \]

\[ E(X) = \sum_{k=1}^{n} k \cdot \binom{n}{k} p^k q^{n-k} \text{ Note: } k = 0 \text{ adds } 0 \text{ to } E(X) \]

\[ = \sum_{k=1}^{n} nk \binom{n-1}{k-1} p^k q^{n-k} \]

\[ = np \cdot \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} q^{n-k} \]

\[ = np \cdot \sum_{j=0}^{n-1} \binom{n-1}{j} p^j q^{n-1-j} \text{ (by shifting index)} \]

\[ = np \cdot (p + q)^n - 1 \text{ (by binomial theorem)} \]

\[ = np \text{ (by } p + q = 1) \]
Linearity

- Expected value of sum of random variables is sum of expected values
- \( E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n) \)
- \( E(aX + b) = a \cdot E(X) + b \)
- Examples:
  - Sum of two cards (with replacement) is \( 14 \frac{8}{13} \)
  - Sum of two die rolls is \( 2 \cdot E(X) \) where \( E(X) = \frac{1}{6} \cdot \sum_{k=1}^{6} k = \frac{21}{6} \) or 7
Complex Example - Expected number of inversions in a polynomial

- A permutation $P$ of integers $1 \ldots n$ is an arrangement of the numbers
- An inversion is where $i < j$ but $j \prec i$ in $P$
- Example: $P = (1, 3, 5, 2, 4)$ the inversions are $(2, 3), (2, 5), (4, 5)$
- Let $l_{i,j}$ be the random variable on the set of all permutations of the first $n$ integers with $l_{i,j} = 1$ if $(i, j)$ is an inversion on the permutation
- For the example, $l_{2,3}(P) = 1, l_{1,4} = 0$
- Let $X$ be the random variable equal to the number of inversions, $X = \sum_{1 \leq i < j \leq n} l_{i,j}$
- For example, $X(P) = 3$
- $E(l_{i,j}) = 1 \times p(l_{i,j} = 1) + 0 \times p(l_{i,j} = 0) = 1/2$ (equally likely inversion as not)
- There are $\binom{n}{2}$ ways for 2 numbers to be arranged out of $n$
- $E(X) = \binom{n}{2} E(l_{i,j}) = \frac{n!}{(n-2)!2*2} = \frac{n*(n-1)}{4}$
Average Case Complexity

- $S$ is the possible inputs to the program
- $X : S \rightarrow \mathbb{R}$, such that $\forall s \in S$, $X(s)$ is the number of operations performed
- Let $p(s)$ be the probability of $s$ being the input to the program
- $\sum_{s \in S} p(s) X(s)$ is the expected (or average) number of operations
Average Complexity Linear Search (text)

- Let $p$ be the probability $x \in A$. Assume $x$ is equally likely to be in any other location
- Counting number of comparisons
- For each element, check to see if at end of array and compare value (2 comparisons per element)
- After loop, one comparison to see if past end of array
- Probability $x$ is at element $k$ is $p/n$
- Probability $x$ is not in list is $q = 1 - p$
- If $x \in A$, then $\sum_{k=1}^{n} \frac{p}{n} (2k + 1) = $
  \[ \frac{p}{n} \sum_{k=1}^{n} (2k + 1) = \frac{p}{n} * (n + 2 \sum_{k=1}^{n} k) \]
  \[ = \frac{p}{n} * (n + 2 \cdot \frac{n(n+1)}{2}) = \frac{p}{n} * (n + n(n+1)) = p(1 + n + 1) = p(n+2) \]
- If $x \not\in A$, then $(2n + 2)q$
- $E(X) = p(n+2) + (2n+2)q$
Variance

- Let $X$ be a random variable on $S$
- Variance on $X$ (denoted $V(X)$) indicates the spread of values in $X(S)$
- $V(X) = \sum_{s \in S} (X(s) - E(X))^2 p(s)$
- Standard deviation $\sigma(X) = \sqrt{V(X)}$
- Example:
  - Blackjack cards: $V(X) = \sum_{s \in S} (X(s) - 7\frac{4}{13})^2 * p(s) =$
  - $(-5\frac{4}{13})^2 * 1/13 + (-4\frac{4}{13})^2 * 1/13 + \ldots + (2\frac{9}{13})^2 * 4/13 + (3\frac{9}{13})^2 * 1/13$
  - $\approx 8.5$
  - $\sigma(X) \approx 2.9$
Variance Continued

- \( V(X) = E(X^2) - E(X)^2 \)
- Example:
  - Blackjack cards: \( E(X)^2 = (7\frac{4}{13})^2 \approx 53.4 \)
  - \( E(X^2) = \frac{1}{13} \times (\sum_{k=2}^{9} k^2 + 121) + \frac{4}{13} \times 100 = \frac{784}{13} \approx 61.9 \)
  - \( \approx 8.5 \)
  - \( \sigma(X) \approx 2.9 \)
- Let \( E(X) = \mu. \) Then \( V(X) = E((X - \mu)^2) \)
  - \( = \frac{1}{13} \times (\sum_{k=2}^{9} (k - 7\frac{4}{13})^2 + (11 - 7\frac{4}{13})^2) + \frac{4}{13} \times (10 - 7\frac{4}{13})^2 = 8.5 \)
- Let \( X \) be a random variable such that \( X(t) = 1 \) if a Bernoulli trial is successful and \( X(t) = 0 \) otherwise.
- Note: Single trial, so \( n = 1 \) for Bernoulli distribution.
  - \( E(X) = p \times 1 + q \times 0 = p. \) \( E(X^2) = p \times 1^2 + q \times 0^2 = p. \)
  - \( E(X)^2 = p^2. \) \( V(X) = p - p^2 = p(1 - p) = pq. \)
Variance Equations

- **Bienayme’s Formula**
  - If $X$ and $Y$ are independent random variables on $S$, then
  $$V(X + Y) = V(X) + V(Y)$$

- **Chebyshev’s Inequality**
  - If $X$ is a random variable on $S$ with probability function $p$, then
  $$p(\lvert X(s) - E(X) \rvert \geq r) \leq \frac{V(X)}{r^2}$$

- **Example:**
  - Probability draw a card 3 or more from the mean of blackjack cards
  - $\frac{V(X)}{r^2} \approx \frac{8.5}{9} \approx 0.94$
  - Actual is 2, 3, 4, $A = \frac{4}{13}$
Section 15

Recurrence Relations
Definitions

- A **recurrence relation** is an equation that expresses $a_n$ in terms of one of more of the previous terms.
- A sequence is a **solution** to a recurrence relation if its terms satisfy the equations
- Examples:
  - Recurrence Relation: $a_n = a_{n-1} + 3, a_0 = 2$. Solution: [2, 5, 8, ...]
  - Fibonacci: $a_n = a_{n-1} + a_{n-2}, a_0 = 1, a_1 = 1$. Solution: [0, 1, 1, 2, 3, 5, ...]
- A **closed form solution** is an equation for each term that does not reference other terms
Linear Homogeneous Recurrence Relation

Definition

- A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form:
  \[ a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \]
  such that every \( c_i \in \mathbb{R} \) and \( c_k \neq 0 \).
- Linear because each right-hand side term is sum of previous terms
- Homogeneous because no terms occur that are not multiples of previous terms
- Constant coefficients means no \( c_i \) can reference \( n \) (but note that 0 is allowed for all but last coefficient)
- The degree is determined by the number of terms required
Linear Homogeneous Recurrence Relation

Examples

- $a_n = \frac{3a_{n-1}}{2}$ is l.h.r.r. of degree 1
- Fibonacci is l.h.r.r of degree 2
- $a_n = 2 \cdot a_{n-5}$ is l.h.r.r of degree 5 (4 terms with 0 as coefficient)
- $a_n = a_{n-1} + 3$ is not l.h.r.r because 3 is not multiple of previous term
- $a_n = 2^n a_{n-1}$ is not l.h.r.r because $2^n$ is not constant coefficient
- $a_n = a_{n-1}^2$ is not l.h.r.r because squared term is not linear
Find closed form equation (of the form $a_n = r^n$).

$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$

$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \ldots + c_k$ – divide both sides by $r^{n-k}$

$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$ – is the characteristic equation

Solutions to the characteristic equation are the characteristic roots

Assuming degree=2 and distinct roots $r_0$ and $r_1$,

$a_n = \alpha_1 r_0^n + \alpha_2 r_1^n$

Use initial terms to solve for $\alpha_1$ and $\alpha_2$
Solving L.H.R.R.

- $a_n = 5a_{n-1} - 6a_{n-2}, a_0 = 1, a_1 = 0$
- $r^n = 5r^{n-1} - 6r^{n-2} \rightarrow r^2 = 5r - 6 \rightarrow r^2 - 5r + 6 = 0$
- Characteristic Roots are 3, 2
- $a_n = \alpha_1 3^n + \alpha_2 2^n$
- $1 = \alpha_1 + \alpha_2 \rightarrow 1 - \alpha_1 = \alpha_2$
- $0 = \alpha_1 * 3 + \alpha_2 * 2 \rightarrow 0 = \alpha_1 * 3 + (1 - \alpha_1) * 2 \rightarrow \alpha_1 = -2 \rightarrow \alpha_2 = 3.$
- $a_n = -2(3^n) + 3(2^n)$
- Check: Sequence solution is $[1, 0, -6, -30, \ldots]$
- $a_3 = -2(3^3) + 3(2^3) = -54 + 24 = -30$
Solving L.H.R.R.

- Fibonacci: \( a_n = a_{n-1} + a_{n-2}, \ a_0 = 0, \ a_1 = 1 \)
- \( r^2 = 1r^1 + 1r^0 \)
- \( r^2 - r^1 - 1 = 0 \)
- Quadratic Equation: \( \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)
- \( a = 1, \ b = -1, \ c = -1, \ \frac{1 + \sqrt{1+4}}{2}, \ \frac{1 - \sqrt{1+4}}{2} \)
- \( a_n = \alpha_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + \alpha_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n \)
- Use \( a_0 \) and \( a_1 \) to determine values for alpha
Solving L.H.R.R. continued

\[
0 = \alpha_1 \left( \frac{1 + \sqrt{5}}{2} \right)^0 + \alpha_2 \left( \frac{1 - \sqrt{5}}{2} \right)^0 = \alpha_1 + \alpha_2
\]

Therefore, \(-\alpha_1 = \alpha_2\)

\[
1 = \alpha_1 \left( \frac{1 + \sqrt{5}}{2} \right)^1 + \alpha_2 \left( \frac{1 - \sqrt{5}}{2} \right)^1
\]

Substituting: \(1 = \alpha_1 \left( \frac{1 + \sqrt{5}}{2} \right)^1 - \alpha_1 \left( \frac{1 - \sqrt{5}}{2} \right)\)

\[
1 = \alpha_1 \left( \frac{1 + \sqrt{5}}{2} - \frac{1 - \sqrt{5}}{2} \right) = \alpha_1 \frac{1 + \sqrt{5} - 1 + \sqrt{5}}{2} = \alpha_1 \sqrt{5}
\]

\[
\alpha_1 = \frac{1}{\sqrt{5}}, \alpha_2 = -\frac{1}{\sqrt{5}}
\]

\[
a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n + \frac{-1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n
\]

Test cases:

\[
\text{fib}(0) = 0. \quad \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^0 + \frac{-1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^0 = \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}} = 0
\]

\[
\text{fib}(5) = 5. \quad \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^5 + \frac{-1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^5 \approx 4.96 - -0.04 = 5
\]

\[
\text{fib}(18) = 2584. \quad \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{18} + \frac{-1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{18} \approx 2584.00007 - (7 * 10^{-5} = 2584
\]
Let $r^2 - c_1 r^1 - c_2 r^0$ have only one real root $x$.

Closed form solution for $a_n = \alpha_1 x^n + \alpha_2 n \cdot x^n$

Example: $a_n = 4a_{n-1} - 4a_{n-2}, a_0 = 6, a_1 = 8$

$r^2 = 4r - 4 \rightarrow r^2 - 4r + 4 = 0$

$\frac{4\pm\sqrt{16-4*1*4}}{2} = \frac{4}{2} = 2$

$a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n$

$a_0 = \alpha_1 2^0 + \alpha_2 0 \cdot 2^0 \rightarrow 6 = \alpha_1$

$a_1 = \alpha_1 2^1 + \alpha_2 1 \cdot 2^1 \rightarrow 8 = 6 \cdot 2 + \alpha_2 \cdot 2 \rightarrow -2 = \alpha_2$

$a_n = 6(2^n) - 2n(2^n)$

Check: $[6, 8, 8, 0, -32, ...]$

$a_4 = 6 \cdot 16 - 8 \cdot 16 = 96 - 128 = -32$
Section 16

Relations
Definitions of Relations

- A relation between sets $A$ and $B$ is a subset of $A \times B$.
- Typically, a relation defines a connection between elements of the set.
- Example 1: $A = \{ \text{students in class} \}$, $B = \{ \text{side of room} \}$, $R(a, b) \leftrightarrow a \text{ sits on } b \text{ side of the room. (on board)}$
- Example 2: $A = \{ \text{volunteers} \}$, $B = \{ \text{food} \}$, $R(a, b) \rightarrow a \text{ likes } b.$ (on board)
- Functions are relations restricted such that elements from $A$ appear only once (Example 1).
- Graphs can show relations.
- Relations can be on one set $A = \{ \text{food} \}$, $B = \{ \text{food} \}$, $R(a, b) \leftrightarrow a \text{ is the same color as } b$ (on board). Usually written as $R(a, a)$.
Properties of Relations

Consider relations on $\mathbb{Z} \times \mathbb{Z}$

- **Reflexive** : $\forall a \in A, R(a, a)$
  - $R(z_0, z_1) \iff z_0 \leq z_1$ is reflexive
  - $R(z_0, z_1) \iff z_0 < z_1$ is NOT reflexive

- **Symmetric** : $\forall a \in A, \forall b \in B, R(a, b) \rightarrow R(b, a)$
  - $R(z_0, z_1) \iff z_0 \leq z_1$ is NOT symmetric
  - $R(z_0, z_1) \iff z_0 = z_1$ is symmetric (and reflexive)
  - $R(z_0, z_1) \iff z_0$ and $z_1$ are relatively prime is symmetric (and not reflexive)

- **Antisymmetric** (poorly named):
  $\forall a \in A, \forall b \in B, R(a, b) \land R(b, a) \rightarrow a = b$
  - $R(z_0, z_1) \iff z_0 \leq z_1$ is antisymmetric
  - $R(z_0, z_1) \iff z_0 = z_1$ is antisymmetric
  - $R(z_0, z_1) \iff z_0$ and $z_1$ are relatively prime is NOT antisymmetric
Properties of Relations

- **Transitive**: \( R(a, b) \land R(b, c) \rightarrow R(a, c) \)
  - \( R(z_0, z_1) \leftrightarrow z_0 \leq z_1 \) is transitive
  - \( R(z_0, z_1) \leftrightarrow z_0 = z_1 \) is transitive
  - \( R(z_0, z_1) \leftrightarrow z_0 \) and \( z_1 \) are relatively prime is NOT transitive
Sets and Relations

- Relations are sets of ordered pairs. Therefore, all set operations apply.

- \( R(z_0, z_1) \iff z_0 \leq z_1 - R(z_0, z_1) \iff z_0 < z_1 = R(z_0, z_1) \iff z_0 = z_1 \)

- Proof:
  - Let \( LEQ = R(z_0, z_1) \iff z_0 \leq z_1 \), \( LT = R(z_0, z_1) \iff z_0 < z_1 \) and \( EQ = R(z_0, z_1) \iff z_0 = z_1 \).
  - Let \((a, b) \in LEQ - LT\). Therefore, \((a, b) \in LEQ\) and \((a, b) \notin LT\).
  - Therefore, \( a \leq b \) and \( a \geq b \) (not less than).
  - Therefore, \( a = b \) and \( EQ(a, b) \). Reverse direction is similar.

- Let \( A = \{1, 2, 3\} \). Let \( R(a, a) = \{(1, 1), (2, 2), (1, 2)\} \) and \( S(a, a) = \{(1, 1), (2, 2), (2, 1)\} \).
  - \( R \cup S = \{(1, 1), (2, 2), (1, 2), (2, 1)\} \)
  - \( R \cap S = \{(1, 1), (2, 2)\} \)
  - \( R \oplus S = \{(1, 2), (2, 1)\} \)
N-ary Relations

- Text is awkward with notation
- Extend notion to n-wise cross product.
- Given sets $S_0, S_1, \ldots, S_{n-1}$, $R \subseteq S_0 \times S_1 \times \ldots \times S_{n-1}$
- $r \in R$ is a $n$-tuple. Note that the ordering is important.
- Relational databases (Oracle, MySQL, SQLServer, etc.) use tables as relations with attributes representing sets
- Example: Students(Id, Name, Major, Favorite Number)
- Collection of attributes is the schema
- Note: Databases allow duplicate elements – database tables are bags of n-tuples
- Id is *primary key* uniquely identifies row in table
Basic Relational Algebra

- Let $R(A, B, C) = \{(1, 2, 3), (2, 3, 4)\}$ and $S(C, D, E) = \{(3, 4, 5), (3, 2, 1)\}$
- $\sigma_P R$ (selection) creates new table with same schema as $R$. A row is in $\sigma_P R$ if it is in $R$ and it satisfies predicate $P$
- $\sigma_{A=1} R = \{(1, 2, 3)\}$
- $\Pi_{A, B} R$ (projection) creates new table with columns $A$ and $B$. There is a 1-1 mapping from each row in $R$ to each row in $\Pi_{A, B} R$
- $\Pi_{B, C} R = \{(2, 3), (3, 4)\}$
- $R \bowtie S$ (natural join) creates new table with union of columns in $R$ and in $S$. A row is in $R \bowtie S$ if $\exists r \in R \land s \in S$ such that $r[R \cap S] = s[R \cap S]$.
- $R \bowtie S = \{(1, 2, 3, 4, 5), (1, 2, 3, 2, 1)\}$
- Note: Results of relational algebra operators are relations. Operations can be composed.
- $\Pi_{A,E}(\sigma_{B=D}(R \bowtie S) = \{(1, 1)\}$