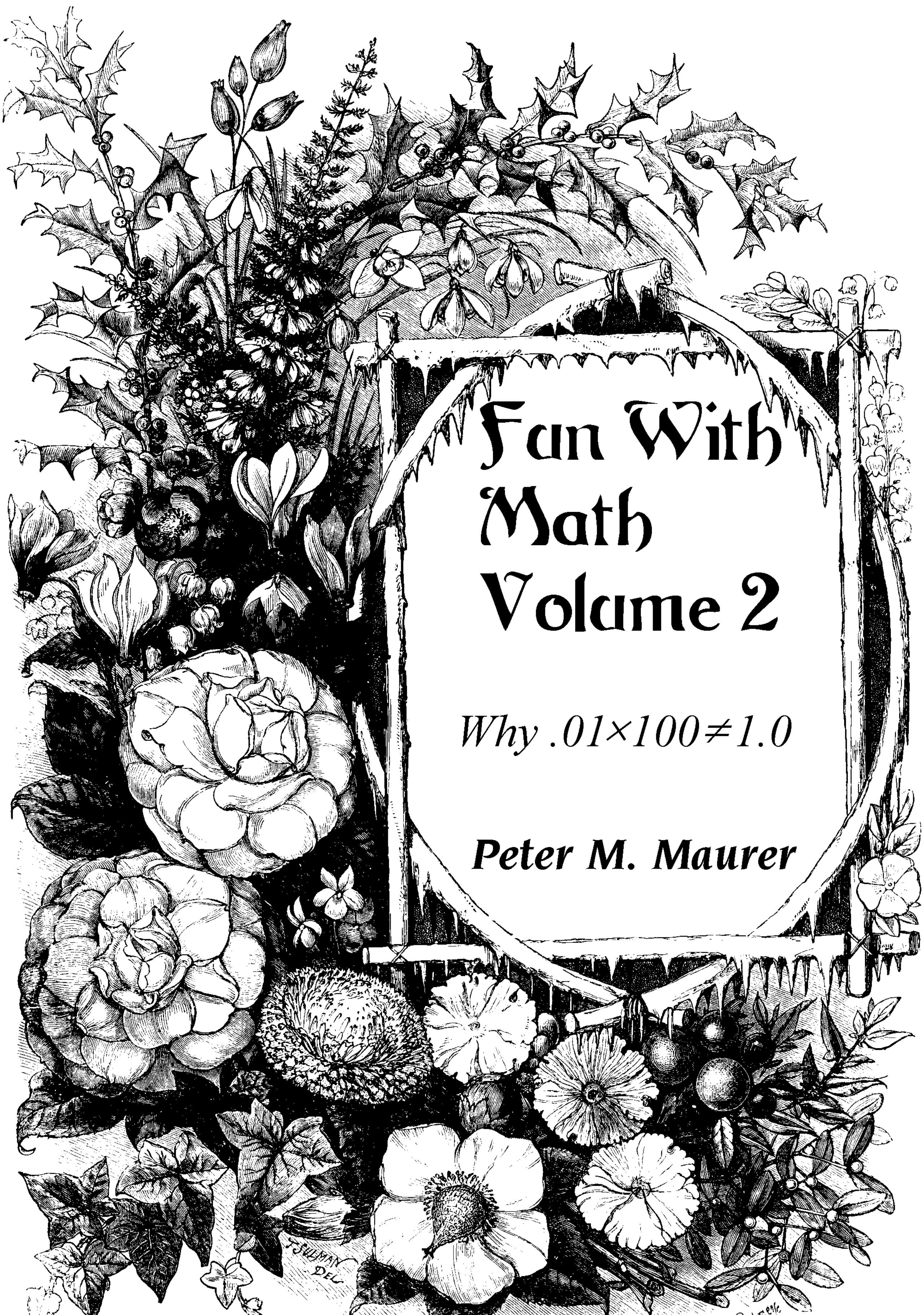

Fun With Math Volume 2

Why .01×100≠1.0

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FUN WITH MATH VOL. II
WHY 100*.01 ≠ 1

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Back in grade school, we all learned that fractions like 1/2 or 1/4 could also be written as decimals. The fraction 1/2 became 0.5, while the fraction 1/4 became 0.25. In fact, this sort of representation seems much neater and concise than the fractional notation, especially when comparing two numbers. After all, when we ask whether 37035/49380 is equal to 196296/261728, (yes) most of us would throw up our hands in despair. And what if the two numbers are not equal? How do we find out which is bigger?

The obvious answer is to divide them both out on your calculator (which will give you .75) and compare the decimal equivalents. If you know something of mathematics you may remember that one can compare a/b with c/d by comparing $a*d$ with $b*c$, but if one has a calculator, why bother?

Although the wonderful world of decimal representations appears to be the solution to all our problems, there is a snake in this "garden of eden," namely repeating decimal expansions. Although one can represent 1/2 and 1/4 quite conveniently in decimal, 1/3 is a problem. The fraction 1/3 has an infinite decimal expansion of .3333333333333... . The fraction 1/7 is even more ill-behaved with a decimal expansion of .142857142857... .

If you've done much arithmetic, you will quickly discover that most fractions have repeating decimal expansions. Given a fraction a/b, it is fairly simple to determine whether the decimal expansion will be finite or infinite. First, reduce a/b to lowest terms, then factor the denominator, b, into its prime factors. If b has any prime factors other than 2 or 5 then the decimal expansion will be infinite, otherwise it will be finite. This is fairly easy to prove, but first we need to pull the statement apart mathematically so we can get a handle on the proof. The above statement can be proven by proving the following two statements.

1. Any finite decimal number can be represented in lowest terms by a fraction a/b where b has no prime factors other than 2 or 5.
2. Given a fraction a/b in lowest terms where b has no prime factors other than 2 or 5, there is a finite decimal representation of a/b.

As is usually the practice, we will prove these statements by describing an algorithm for finding the fraction given the decimal, and an algorithm for finding the decimal given the fraction. Starting with a decimal, d, assume that there are no significant digits to the left of the decimal point and k significant digits to the right. If we multiply d by 10^k, the result will be a whole number. Since $d*10^k$ is a whole number, we can use it as the numerator of a fraction. The number 10^k is also a whole number, so we can use it as the denominator of a fraction giving us the following.

$$f = \frac{d*10^k}{10^k}(=d)$$

Of course, *f* may not be in lowest terms, but if it isn't we can put it in lowest terms by the following procedure. First factor the numerator and the denominator into prime factors. Then cancel prime factors from the numerator and denominator, until no more cancellations are possible. Finally, multiply out the existing prime factors to obtain the lowest-terms result. Since this procedure does not create new prime factors in the denominator, and since 10^k has no prime factors other than 2 or 5, the fraction *f* expressed in lowest terms has no prime factors other than 2 or 5 in the denominator.

Now, suppose we are given a fraction *a*/*b* in lowest terms with $a<b$ and assume that *b* has no prime factors other than 2 or 5. If this is the case then *b* can be written as 2^i5^j, where *i* and *j* are both greater than or equal to zero. Let $k=\max(i,j)$ and let $x=10^k/b$. Since $10^k = 2^k5^k$, and since *k* is the maximum of *i* and *j*, *x* must be a whole number. Now compute the new fraction:

$$f = \frac{ax}{bx} = \frac{ax}{10^k}$$

We can get rid of the denominator in *f* by moving the decimal point in *ax* (which is at the extreme right) *k* positions to the left. This gives a finite decimal expansion which is equal to the original number *a*/*b*.

Now, the reason that 2 and 5 are the critical prime factors is because 10=2*5. In other words, 2 and 5 are the prime factors of the radix. We can extend our ideas to bases other than 10 by observing that a number *a*/*b* in lowest terms will have a finite base-*k* expansion if and only if all prime factors of *b* are also prime factors of *k*.

This fact was noticed many years ago by the founders of the Duo-Decimal Society, which advocated replacing the decimal system by a duo-decimal number system based on the number 12. (Electronic calculators seem to have killed off the Duo-Decimal Society.) To join the Duo-Decimal Society one had to demonstrate a proficiency in base-12 arithmetic. (And presumably subscribe to the high-ideals of the society.) The reason for advocating base-12 arithmetic seems to be the fact that 12 has more factors than 10, although the shrewd observer will note that the number of prime factors is the same. In base 12, the expansion of 1/3 is finite, as are the expansions of 1/2, 1/4, 1/6, 1/8, and 1/9. (1/3=0.4, 1/2=0.6, 1/4=0.3, 1/6=0.2, 1/8=0.16, and 1/9=0.14) The expansions of (1/5, 1/7, 1/10, and 1/11) are infinite. Oh, excuse me! The Duo-Decimal Society insists that ten be represented by **T** and eleven be represented by **E**. (This might be confusing to your floating-point conversion routine, but some sacrifices must be made.) Actually in its later years, the radical elements of the society decided that **T** and **E** were too "decimal oriented." Since they wanted to reserve the word "ten" for the number 10 (actually 12 as we know it) they decided that **T**(ten) and **E**(eleven) were inappropriate symbols for their number system. They substituted the new words *elf* and *kel* for the numbers ten and eleven, and (as I recall) used the symbols *e* and *k* to represent them. (A bit confusing if you are working with natural logs.) The final issue of their journal included a little song about converting from decimal to duo-decimal. The only line I remember goes something like "And next comes ten which out we throw." To get back to the point, in duo-decimal

1/5=0.294729472947..., 1/7 = 0.186**T**35186**T**35..., 1/**T** = 0.124972497... and 1/**E** = 0.111111... . In decimal one half of the denominators from 2-9 are repeating decimals, while in duo-decimal only 2/5 of them are. This gives a 1/10 (or should I say 1/**T**) advantage to the duo decimal system. Why anyone would consider this important is beyond me.

When this principle is applied to binary numbers, particularly to the floating point representation of fractions, some interesting facts come to light. In binary, fraction a/b in lowest terms, has a finite binary expansion only if b is a power of 2. Since neither 1/10 or 1/100 are powers of 2, neither 1/10 nor 1/100 can be represented exactly in binary. To be precise, 1/10 = 0.000110011..., and 1/100 = .0.00000101000111101 0111... (the repeating group is 00001010001111010111). Therefore, when you write a program such as the following, you will probably get the answer "They are different." The adoption of the IEEE floating point standard has made it much more difficult to generate these sorts of anomalies, so you might have to go to 1000*0.001 or 10000*0.0001 to get a discrepancy.

```
#include <stdio.h>
#include <stdlib.h>

void main(argc,argv)
int argc;
char *argv[];
{
   int i;
   float x;

   x = 0.0;
   for (i=0 ; i<100 ; i++)
   {
      x = x + 0.01;
   }
   if (x == 1.0)
   {
      fprintf(stdout,"They are equal");
   }
   else
   {
      fprintf(stdout,"They are different");
   }
}
```

Curiously enough, the rules for converting numbers from binary to decimal and back are different between fractions and whole numbers. Binary notation requires more digits than decimal notation to represent a whole number. The ratio can be worked out like this. If a number is less than 10^k then it can be represented using k decimal digits. If a number is less than 2^j, then it can be represented using j binary digits. The log base two of 2^j is j and the log base 10 of 10^k is k. The number of digits required to represent a number n in base b is related to the log base b of n. (It is, in fact, equal to $\lceil \log_b(n+1) \rceil$). To determine the difference in the size of a number represented in two different bases, it is necessary to obtain the ratio of the logarithms of the number:

$$\frac{\log_2 n}{\log_{10} n} = \frac{\log_2 n}{\left(\dfrac{\log_2 n}{\log_2 10}\right)} = \log_2 10 = \frac{\log_x 10}{\log_x 2} = \frac{1}{\log_{10} 2} \approx 3.322$$

For exact representations of decimals, no such ratio exists, because some finite decimal expansions are infinite in binary. Converting the other way is easier, since any number that has a finite binary expansion also has a finite decimal expansion. The surprising fact is, however, that any finite binary expansion requiring *k* binary digits will require *k* decimal digits for exact representation. This stems from the fact that 2 divides both radixes, but 2^2 divides neither. If the representations are not required to be exact, then the 3.322 ratio applies. In other words, if two numbers are allowed to be represented in both binary and decimal with the same amount of imprecision, then the binary expansion will require 3.322 more digits than the decimal expansion, on the average.

If you want to experiment with generating base-*k* expansions, this is the program used to get the expansions mentioned above.

```
#include <stdio.h>
#include <stdlib.h>

void main(argc,argv)
int argc;
char *argv[];
{
   int x,y,i;

   x = 7;   /* the number to be expanded as 1/x */
   y = 10;  /* 10 is the radix, substitute 2 for binary */
   for (i=0 ; i<100 ; i++)  /* 100 is the number of digits */
   {
      putc(y/x + '0',stdout);
      y = (y % x) * 10;/* 10 is the radix, substitute 2 for binary */
   }
}
```

One additional point that should be mentioned is the length of the repeating section of the expansion. Let us confine ourselves to decimal expansions of numbers of the form 1/*b* where *b* and 10 are relatively prime (share no factors). The length of the expansion is determined by the solutions *k* to the equation $10^k \equiv 1 \bmod b$. If *k* is the smallest integer solution to this equation, then the decimal expansion of 1/*b* will have a repeating section with *k* digits. The solutions of this equation are at a maximum when *b* is a prime number. If *b* is a prime number it is possible for the repeating section to have as many as *b*-1 digits. Since number theory tells us that $a^{b-1} \equiv 1 \bmod b$, regardless of the values of *a* and *b*, the maximum length for the repeating section of any expansion of the form 1/*b* is *b*-1.

Some expansions of 1/*b* have a repeating section and a non-repeating section. Obviously, the non-repeating section precedes the repeating section. The length of the non-repeating section can be found by factoring the number into its prime factors. If the number has prime factors of 2 or 5 or both, then it will have a non-repeating section. Let *i* be the largest integer such that 2^i divides *b* evenly, and *j* be the largest integer such that 5^j divides *b* evenly. The length of the non-repeating section will be equal to the maximum of *i* and *j*. The length of the repeating section will be identical to the length of the repeating section of the decimal expansion of the following fraction.

$$\frac{1}{\left(\frac{b}{2^i 5^j}\right)}$$